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LETTER TO THE EDITOR

Jordanian twists on deformed carrier subspaces

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Abstract. The nontrivial subspaces with primitive coproducts are found in the deformed universal enveloping algebras. They can form carrier subspaces for additional Jordanian twists. The latter can be used to construct sequences of twists for algebras whose root systems contain long series of roots. The corresponding twist for the $so(5)$ algebra is given explicitly.

1. Introduction

The triangular Hopf algebras $\mathcal{A}(m, \Delta, S, \eta, \epsilon; \mathcal{R})$ with the R -matrix satisfying the unitarity condition $\mathcal{R}_{21}\mathcal{R} = 1$ form a subclass of quasitriangular Hopf algebras [1]. Many of them can be considered as quantizations of triangular Lie bialgebras L with antisymmetric classical r -matrices $r = -r_{21}$ satisfying the classical Yang–Baxter equation. The quantization is defined by a twisting element $\mathcal{F} = \sum f_{(1)} \otimes f_{(2)} \in \mathcal{A} \otimes \mathcal{A}$ with an expansion $\mathcal{F} = 1 + \frac{1}{2}hr + \dots$ [2]. The terms of this decomposition were defined in [2] using the BCH series related to the central extension of the r -matrix carrier Lie algebra L . In applications the knowledge of the twisting element is highly desirable, giving the (twisted) \mathcal{R} -matrix $\mathcal{R}_{\mathcal{F}} = \mathcal{F}_{21}\mathcal{R}\mathcal{F}^{-1}$ and twisted coproduct $\Delta_{\mathcal{F}} = \mathcal{F}\Delta\mathcal{F}^{-1}$. The explicit expressions of the twist elements \mathcal{F} were found [3, 4], for the carrier algebras L with the special properties of their triangular decompositions. The constructions (chains of extended Jordanian twists) proposed there for the higher-dimensional carriers L were based on the effect of primitivization of the carrier subalgebras $L' \subset L$ for certain twists (the full canonically extended twists) performed in L . In this letter we present the other possibility of composing the Jordanian and extended Jordanian twists. We show that under certain conditions when all the coproducts in L are nontrivially twisted there exists in $U(L)$ the deformed primitive carrier subspace $L'_{\mathcal{F}}$. Thus the twist \mathcal{F} can be composed with the next one \mathcal{F}' defined on $L'_{\mathcal{F}}$. This effect greatly enlarges the possible applications of extended twists and chains. In a special form the established sequences of twists can be applied to deform the universal enveloping algebras for the classical series B_N and C_N . An interesting question of connections of constructed twists with recently found quasi-Hopf twistings [5, 6] will be discussed in further publications.

We hope that the constructed twisting elements for the orthogonal Lie algebras will describe the important deformations for the now actively studied anti-de-Sitter field theories (cf, e.g., [7]) and the quantization of coboundary bialgebra structures of conformal algebras, described by triangular classical r -matrices (cf, e.g., [8]).

2. Extended twists and the primitivity of deformed carrier subspaces

Let g be a Lie algebra with the root system Λ_g , generators $L_\lambda, \lambda \in \Lambda_g$, Cartan generators $H_\alpha, \alpha \in S_{\Lambda_g}$ and the universal enveloping algebra $U(g)$. Consider the class \mathcal{B} of two-dimensional Borel subalgebras in $U(g)$ associated with a fixed Cartan generator H and

$$E_{\mathcal{B}} = E + \xi B_i B^i$$

where $E \equiv L_{\lambda_0}$ has the canonically normalized root λ_0 dual to H , $[H, E] = E$ and $\{B_i, B^j\}$ is the set of eigenvectors for H with the property

$$\begin{aligned} [H, B_i] &= \beta_i B_i \\ [H, B^j] &= \beta^j B^j \quad \beta_i + \beta^i = 1. \end{aligned} \tag{1}$$

This guarantees that $[H, E_{\mathcal{B}}] = E_{\mathcal{B}}$.

Let us consider the conditions sufficient for the primitivity of the generators in \mathcal{B} . For the generator $E_{\mathcal{B}}$ the situation obviously depends on the value of ξ :

- (I) $\xi = 0$, in particular this is the case of the undeformed $U(g)$ where the subalgebras \mathcal{B} are primitive;
- (II) $\xi \neq 0$, this case can be realized only in the deformed $U(g)$.

The primitivity of H depends on the injection of the carrier subalgebras $L_i^{(4)}$ of the preceding extended Jordanian twists (see [3]) into the carrier algebra of the final composition of twists. Consider the Borel subalgebra $B(2) \subset L^{(4)}$ generated by one of the generators $L_{\lambda_0^\perp}$ with the property $[H_{\lambda_0^\perp}, L_{\lambda_0^\perp}] = L_{\lambda_0^\perp}$. Let $\Phi_{\mathcal{J}_\perp}$ be the Jordanian twist based on this $B(2)$, i.e.

$$\Phi_{\mathcal{J}_\perp} = \exp\{H_{\lambda_0^\perp} \otimes \sigma_0^\perp\} \quad \sigma_0^\perp = \ln(1 + L_{\lambda_0^\perp}). \tag{2}$$

From now on we shall use the notation and the normalization of the structure constants introduced in [4]. Let π_\perp be the set of the constituent roots for λ_0^\perp :

$$\pi_\perp = \{\lambda', \lambda'' | \lambda' + \lambda'' = \lambda_0^\perp; \lambda' + \lambda_0^\perp, \lambda'' + \lambda_0^\perp \notin \Lambda_g\}. \tag{3}$$

For any $\lambda' \in \pi_\perp$ there must be such an element $\lambda'' \in \pi_\perp$ that $\lambda' + \lambda'' = \lambda_0^\perp$. So, π_\perp is naturally decomposed as

$$\pi_\perp = \pi'_\perp \cup \pi''_\perp \quad \pi'_\perp = \{\lambda'\} \quad \pi''_\perp = \{\lambda''\}. \tag{4}$$

Consider now the sequence of extensions $\Phi_{\mathcal{E}_\perp}$ for the Jordanian twist $\Phi_{\mathcal{J}_\perp}$:

$$\Phi_{\mathcal{E}_\perp} = \prod_{\lambda' \in \pi'_\perp} \Phi_{\mathcal{E}_{\lambda'}} = \prod_{\lambda' \in \pi'_\perp} \exp\{L_{\lambda'} \otimes L_{\lambda_0^\perp - \lambda'} e^{-\frac{1}{2}\sigma_0^\perp}\}. \tag{5}$$

The complete sequence of extended twists looks like

$$\mathcal{F}_{\mathcal{B}_\perp} = \Phi_{\mathcal{E}_\perp} \Phi_{\mathcal{J}_\perp}. \tag{6}$$

In this situation the primitivity of H is guaranteed by the fact that H^* is orthogonal to the only root composed by the elements of π_\perp . Let us pass to the twisted coproduct for the generator $E_{\mathcal{B}}$:

$$\Delta_{\mathcal{F}_{\mathcal{B}_\perp}}(E_{\mathcal{B}}) = \Delta_{\mathcal{F}_{\mathcal{B}_\perp}}(E) + \xi \Delta_{\mathcal{F}_{\mathcal{B}_\perp}}(B_i) \Delta_{\mathcal{F}_{\mathcal{B}_\perp}}(B^i) = \Delta_{\Phi_{\mathcal{E}_\perp}}(E) + \xi \Delta_{\mathcal{F}_{\mathcal{B}_\perp}}(B_i) \Delta_{\mathcal{F}_{\mathcal{B}_\perp}}(B^i) \tag{7}$$

$$\Delta_{\Phi_{\mathcal{E}_\perp}}(E) = \Phi_{\mathcal{E}_\perp}(E \otimes 1 + 1 \otimes E)(\Phi_{\mathcal{E}_\perp})^{-1}. \tag{8}$$

We have the following possibilities.

- (a) $\lambda' + \lambda_0, \lambda_0^\perp - \lambda' + \lambda_0$ are not in Λ_g . In this case E remains primitive and this is just case I.
- (b) $\lambda' + \lambda_0, \lambda_0^\perp - \lambda' + \lambda_0$ are in Λ_g but not all of them are in π_\perp . In this situation the carrier subalgebra is to be enlarged. For instance these roots may be in the other links of the chain of twists. We shall consider this important possibility elsewhere.

(c) $\lambda' + \lambda_0, \lambda_0^\perp - \lambda' + \lambda_0$ are in π_\perp . Let us concentrate our attention on this case.

When the tensor $C = \sum_{\lambda'} (L_{\lambda'} \otimes L_{\lambda_0^\perp - \lambda'} e^{-\frac{1}{2}\sigma_0^\perp})$ is an invariant of the generic $B(2)$ subalgebra (with $[H, E] = E$),

$$[C, \Delta^{\text{prim}}(H)] = [C, \Delta^{\text{prim}}(E)] = 0$$

we have the Matreshka effect [4] and are again in situation I.

Let us pass to the case of a noninvariant C . Moreover, we shall suppose that the subspaces π'_\perp and π''_\perp are not conserved by the shift with the root λ_0 and impose three additional conditions: (i) $\lambda' + \lambda_0 \in \pi''_\perp$; (ii) $\lambda' + \lambda_0^\perp$ and $\lambda'' + \lambda_0^\perp$ are not in Λ_g ; and (iii) all the λ_0 -series of the roots are short. The minimal subset involved in this action contains two roots in π'_\perp : λ' and

$$\tilde{\lambda}' = -\lambda' - \lambda_0 + \lambda_0^\perp. \tag{9}$$

This means that it is sufficient to regard the factors $\{B_i, B^j\}$ as depending on the list of generators:

$$L_{\lambda'}, \quad L_{\lambda_0^\perp - \lambda'}, \quad L_{\tilde{\lambda}'}, \quad L_{\lambda_0^\perp - \tilde{\lambda}'}, \quad L_{\lambda_0^\perp} \quad \text{and} \quad L_{\lambda_0}. \tag{10}$$

The twisted coproducts $\Delta_{\mathcal{F}_{B_\perp}}$ for the first five of them are known (see [4]). In particular, σ_0^\perp is primitive and

$$\begin{aligned} \Delta_{\mathcal{F}_{B_\perp}}(L_{\lambda'; \tilde{\lambda}'}) &= L_{\lambda'; \tilde{\lambda}'} \otimes e^{-\frac{1}{2}\sigma_0^\perp} + 1 \otimes L_{\lambda'; \tilde{\lambda}'} \\ \Delta_{\mathcal{F}_{B_\perp}}(L_{\lambda_0^\perp - \lambda'; \lambda_0^\perp - \tilde{\lambda}'}) &= L_{\lambda_0^\perp - \lambda'; \lambda_0^\perp - \tilde{\lambda}'} \otimes e^{+\frac{1}{2}\sigma_0^\perp} + e^{\sigma_0^\perp} \otimes L_{\lambda_0^\perp - \lambda'; \lambda_0^\perp - \tilde{\lambda}'}. \end{aligned} \tag{11}$$

Returning to expression (8) we obtain the last coproduct

$$\begin{aligned} \Delta_{\mathcal{F}_{B_\perp}}(E) &= \prod_{\lambda', \tilde{\lambda}'} \exp\{L_{\lambda'} \otimes L_{\lambda_0^\perp - \lambda'} e^{-\frac{1}{2}\sigma_0^\perp}\} (E \otimes 1 + 1 \otimes E) \exp\{-L_{\lambda'} \otimes L_{\lambda_0^\perp - \lambda'} e^{-\frac{1}{2}\sigma_0^\perp}\} \\ &= E \otimes 1 + 1 \otimes E + C_{\lambda' \lambda_0}^{\lambda_0^\perp - \tilde{\lambda}'} L_{\lambda_0^\perp - \tilde{\lambda}'} \otimes L_{\lambda_0^\perp - \lambda'} e^{-\frac{1}{2}\sigma_0^\perp} \\ &\quad + C_{\lambda' \lambda_0}^{\lambda_0^\perp - \tilde{\lambda}'} L_{\lambda_0^\perp} \otimes L_{\lambda_0^\perp - \tilde{\lambda}'} L_{\lambda_0^\perp - \lambda'} e^{-\sigma_0^\perp} + C_{\tilde{\lambda}' \lambda_0}^{\lambda_0^\perp - \lambda'} L_{\lambda_0^\perp - \lambda'} \otimes L_{\lambda_0^\perp - \tilde{\lambda}'} e^{-\frac{1}{2}\sigma_0^\perp}. \end{aligned} \tag{12}$$

We are still free in the normalization of E ; we fix it so that $C_{\lambda' \lambda_0}^{\lambda_0^\perp - \tilde{\lambda}'} = -1$. Notice that the condition $\lambda' + \lambda_0 = \lambda_0^\perp - \tilde{\lambda}'$ means also that we obtain a closed subalgebra. According to the normalization of the $L^{(4)}$ structure constants mentioned above this gives the following value for the last remaining constant:

$$C_{\lambda' \lambda_0}^{\lambda_0^\perp - \tilde{\lambda}'} C_{\lambda_0^\perp - \tilde{\lambda}' \lambda'}^{\lambda_0^\perp} + \sum_{\nu} C_{\lambda' \lambda'}^{\nu} C_{\nu \lambda_0}^{\lambda_0^\perp} + C_{\lambda_0 \lambda'}^{\lambda_0^\perp - \lambda'} C_{\lambda_0^\perp - \lambda' \lambda'}^{\lambda_0^\perp} = 0 \Big\} \Rightarrow C_{\tilde{\lambda}' \lambda_0}^{\lambda_0^\perp - \lambda'} = -1.$$

The final expression for the coproduct $\Delta_{\mathcal{F}_{B_\perp}}(E)$ is

$$\begin{aligned} \Delta_{\mathcal{F}_{B_\perp}}(E) &= E \otimes 1 + 1 \otimes (E + L_{\lambda_0^\perp - \tilde{\lambda}'} L_{\lambda_0^\perp - \lambda'} e^{-\sigma_0^\perp}) \\ &\quad - (L_{\lambda_0^\perp - \tilde{\lambda}'} \otimes L_{\lambda_0^\perp - \lambda'} + L_{\lambda_0^\perp - \lambda'} \otimes L_{\lambda_0^\perp - \tilde{\lambda}'}) (1 \otimes e^{-\frac{1}{2}\sigma_0^\perp}) - e^{\sigma_0^\perp} \otimes L_{\lambda_0^\perp - \tilde{\lambda}'} L_{\lambda_0^\perp - \lambda'} e^{-\sigma_0^\perp}. \end{aligned}$$

Thus for the factors $\{B_i, B^j\}$ we have the following (coalgebraic) equation:

$$\begin{aligned} 1 \otimes (L_{\lambda_0^\perp - \tilde{\lambda}'} L_{\lambda_0^\perp - \lambda'} e^{-\sigma_0^\perp}) - (L_{\lambda_0^\perp - \tilde{\lambda}'} \otimes L_{\lambda_0^\perp - \lambda'} + L_{\lambda_0^\perp - \lambda'} \otimes L_{\lambda_0^\perp - \tilde{\lambda}'}) (1 \otimes e^{-\frac{1}{2}\sigma_0^\perp}) \\ - e^{\sigma_0^\perp} \otimes L_{\lambda_0^\perp - \tilde{\lambda}'} L_{\lambda_0^\perp - \lambda'} e^{-\sigma_0^\perp} + \xi \Delta_{\mathcal{F}_{B_\perp}}(B_i) \Delta_{\mathcal{F}_{B_\perp}}(B^i) \\ = (\xi B_i B^i) \otimes 1 + 1 \otimes (\xi B_i B^i). \end{aligned} \tag{13}$$

This demonstrates that the list of realizations for B in (10) might be reduced to the set

$$L_{\lambda_0^\perp - \tilde{\lambda}'}, \quad L_{\lambda_0^\perp - \lambda'} \quad \text{and} \quad L_{\lambda_0^\perp}.$$

This immediately leads to the solution of the equations (11) and (13) that gives the following answer:

$$E_B = E + L_{\lambda_0^\perp - \tilde{\lambda}'} L_{\lambda_0^\perp - \lambda'} e^{-\sigma_0^\perp} \quad \xi = 1. \tag{14}$$

We have proved that in the twisted universal enveloping algebra $U_{\mathcal{F}_{B_\perp}}$ one can find the deformed primitive Borel subspace. In case I this was one of the basic effects in constructing chains of extended twists [4]. In case II this also provides the possibility to compose new twists such as

$$\mathcal{F}_{B\mathcal{J}\mathcal{E}} = \Phi_{B\mathcal{J}} \Phi_{\mathcal{E}_\perp} \Phi_{\mathcal{J}_\perp} \tag{15}$$

where the second Jordanian factor is defined on the deformed carrier subspace generated by $\{H, E_B\}$,

$$\begin{aligned} \Phi_{B\mathcal{J}} &= \exp(H \otimes \sigma(E_B)) \\ \sigma(E_B) &= \ln(1 + E_B). \end{aligned} \tag{16}$$

The carrier algebra for the twists $\mathcal{F}_{B\mathcal{J}\mathcal{E}}$ is eight dimensional with the generators

$$\{H, H_{\lambda_0^\perp}, L_{\lambda_0}, L_{\lambda_0^\perp}, L_{\lambda'}, L_{\lambda_0^\perp - \lambda'}, L_{\tilde{\lambda}'}, L_{\lambda_0^\perp - \tilde{\lambda}'}\} \tag{17}$$

and the set of roots

$$\lambda_0 \quad \lambda_0^\perp \quad \lambda' \quad \lambda_0^\perp - \lambda' \quad \tilde{\lambda}' \quad \lambda_0^\perp - \tilde{\lambda}'. \tag{18}$$

It is necessary to distinguish the following special case of the general structure considered above. This is the case when in the root condition (9) the roots λ' and $\tilde{\lambda}'$ coincide:

$$\lambda' + \lambda_0 = \lambda_0^\perp - \lambda'. \tag{19}$$

The root subset now contains four roots:

$$\lambda_0^\perp \quad \lambda_0 \quad \lambda' \quad \lambda_0^\perp - \lambda' \tag{20}$$

and we have a long λ' -series of the root λ_0^\perp :

$$\lambda_0^\perp \quad \lambda_0^\perp - \lambda' \quad \lambda_0^\perp - 2\lambda' \in \Lambda_g. \tag{21}$$

This is obviously the property characteristic for the series B_N and C_N of simple Lie algebras and the exceptional algebra F_4 . In such deformations the twisting elements are still of the form (15) and (16) while the expression (14) for E_B is to be substituted by

$$E_{B\mathcal{O}} = E + \frac{1}{2}(L_{\lambda_0^\perp - \lambda'})^2 e^{-\sigma_0^\perp}.$$

3. Examples

Identifying the roots (18) or (20) of the carrier subalgebras with the root subsets in simple Lie algebras we can perform new twisting deformations for them specific for the properties of their root systems.

In the case of the eight-dimensional algebra L the minimal simple algebra where the effect described above can be illustrated is the Lie algebra $U(sl(4))$. The elements (17) can be identified with the following generators of $sl(4)$:

$$\begin{aligned} H &= H_{23} & H_{\lambda_0^\perp} &= H_{14} & L_{\lambda_0} &= E_{23} & L_{\lambda_0^\perp} &= E_{14} \\ L_{\lambda'} &= E_{12} & L_{\lambda_0^\perp - \lambda'} &= E_{24} & L_{\tilde{\lambda}'} &= -E_{34} & L_{\lambda_0^\perp - \tilde{\lambda}'} &= E_{13}. \end{aligned}$$

After the first (minimal) extended twist [3]

$$\Phi_{\mathcal{E}} \Phi_{\mathcal{J}_\perp} = e^{(E_{12} \otimes E_{24} e^{-\frac{1}{2}\sigma_{14}})} e^{(H_{14} \otimes \sigma_{14})} \quad \sigma_{14} = \ln(1 + E_{14})$$

the second extension factor can have the form

$$\Phi_{\mathcal{E}'} = \exp(-E_{34} \otimes E_{13} e^{-\frac{1}{2}\sigma_{14}}).$$

According to formula (14) after the sequence of extended twists $\Phi_{\mathcal{E}'}\Phi_{\mathcal{E}}\Phi_{\mathcal{J}_\perp}$ the deformed carrier subspace of primitive elements forms the Borel subalgebra $\mathcal{B} = \{H_{23}, E_{\mathcal{B}}\}E_{\mathcal{B}} \equiv -E_{23} + E_{13}E_{24}e^{-\sigma_{14}}$. So it is possible to use $\sigma_{\mathcal{B}} = \ln(1 + E_{\mathcal{B}})$ and apply additionally the twist (see (16)) $\Phi_{\mathcal{B}\mathcal{J}} = \exp(H_{23} \otimes \sigma_{\mathcal{B}})$ to the deformed algebra $U_{\mathcal{E}'\mathcal{E}\mathcal{J}}(sl(4))$. In this case the final twisting element looks like

$$\mathcal{F}_{\mathcal{B}\mathcal{J}\mathcal{E}'\mathcal{E}\mathcal{J}} = \Phi_{\mathcal{B}\mathcal{J}}\Phi_{\mathcal{E}'}\Phi_{\mathcal{E}}\Phi_{\mathcal{J}_\perp}. \tag{22}$$

We shall consider such cases in full detail in the forthcoming publications.

The minimal simple Lie algebra that has the root subset (20) is $so(5)$. When the root system of $so(2N + 1)$ is fixed in the standard e -basis as

$$\Lambda_{so(2N+1)} = \{\pm e_i, \pm e_i \pm e_j | i, j = 1, \dots, N\}$$

then in accordance with the property (19) the set (20) can be injected into $\Lambda_{so(5)}$ as follows:

$$\lambda_0^\perp = e_1 + e_2 \quad \lambda_0 = e_1 - e_2 \quad \lambda' = e_2 \quad \lambda_0^\perp - \lambda' = e_1.$$

Thus we obtain the six-dimensional subalgebra $L^{(6)} \subset so(5)$ generated by the set

$$\{H_{1+2}, E_{1+2}, H_{1-2}, E_{1-2}, E_1, E_2\}.$$

In terms of the ordinary antisymmetric Okubo matrices M_{ij} the following list of generators in the defining representation $d(L^{(6)})$:

$$\begin{aligned} d(H_{\lambda_0^\perp}) &= d(H_{1+2}) = -\frac{i}{2}(M_{12} + M_{34}) \\ d(H) &= d(H_{1-2}) = -\frac{i}{2}(M_{12} - M_{34}) \\ d(L_{\lambda_0^\perp}) &= d(E_{1+2}) = \frac{1}{2}(-M_{24} + iM_{23} + iM_{14} + M_{13}) \\ d(L_{\lambda_0}) &= d(E_{1-2}) = \frac{1}{2}(-M_{24} - iM_{23} + iM_{14} - M_{13}) \\ d(L_{\lambda_0^\perp - \lambda'}) &= d(E_1) = \frac{1}{\sqrt{2}}(M_{25} - iM_{15}) \\ d(L_{\lambda'}) &= d(-E_2) = \frac{1}{\sqrt{2}}(-M_{45} + iM_{35}) \end{aligned}$$

fits the normalization conditions for $L^{(6)}$.

The canonical extended twist $\mathcal{F}_{\mathcal{E}\mathcal{J}}$ based on $L^{(4)}$ with the generators $\{H_{1+2}, E_{1+2}, E_1, -E_2\}$,

$$\mathcal{F}_{\mathcal{E}\mathcal{J}} = \exp(-E_2 \otimes E_1 e^{-\frac{1}{2}\sigma_{1+2}}) \exp(H_{1+2} \otimes \sigma_{1+2}) \quad \sigma_{1+2} = \ln(1 + E_{1+2})$$

leads to the deformed algebra $U_{\mathcal{E}\mathcal{J}}(L^{(6)})$ with the coproducts

$$\begin{aligned} \Delta_{\mathcal{E}\mathcal{J}}(H_{1+2}) &= H_{1+2} \otimes e^{-\sigma_{1+2}} + 1 \otimes H_{1+2} + E_2 \otimes E_1 e^{-\frac{3}{2}\sigma_{1+2}} \\ \Delta_{\mathcal{E}\mathcal{J}}(E_{1+2}) &= E_{1+2} \otimes e^{\sigma_{1+2}} + 1 \otimes E_{1+2} \\ \Delta_{\mathcal{E}\mathcal{J}}(E_2) &= E_2 \otimes e^{-\frac{1}{2}\sigma_{1+2}} + 1 \otimes E_2 \\ \Delta_{\mathcal{E}\mathcal{J}}(E_1) &= E_1 \otimes e^{\frac{1}{2}\sigma_{1+2}} + e^{\sigma_{1+2}} \otimes E_1 \\ \Delta_{\mathcal{E}\mathcal{J}}(H_{1-2}) &= H_{1-2} \otimes 1 + 1 \otimes H_{1-2} \\ \Delta_{\mathcal{E}\mathcal{J}}(E_{1-2}) &= E_{1-2} \otimes 1 + 1 \otimes E_{1-2} - E_1 \otimes E_1 e^{-\frac{1}{2}\sigma_{1+2}} - \frac{1}{2}E_{1+2} \otimes E_1^2 e^{-\sigma_{1+2}}. \end{aligned}$$

According to the arguments presented in section 2 we have in $U_{\mathcal{E}\mathcal{J}}(\mathbf{L}^{(6)})$ the primitive subalgebra $\mathcal{B} = \{H_{1-2}, E_{\mathcal{B}\mathcal{O}}\}$ on the deformed subspace with

$$E_{\mathcal{B}\mathcal{O}} = E_{1-2} + \frac{1}{2}E_1^2e^{-\sigma_{1+2}}.$$

In this case the ‘shifted’ Jordanian factor (see (16))

$$\begin{aligned} \Phi_{\mathcal{B}\mathcal{J}} &= \exp(H \otimes \sigma_{\mathcal{B}\mathcal{O}}) = \exp(H_{1-2} \otimes \sigma_{\mathcal{B}\mathcal{O}}) \\ \sigma_{\mathcal{B}\mathcal{O}} &= \ln(1 + E_{\mathcal{B}\mathcal{O}}) \end{aligned}$$

can be applied to $U_{\mathcal{E}\mathcal{J}}(\mathbf{L}^{(6)})$ (and/or to $U_{\mathcal{E}\mathcal{J}}(so(5))$). The result will be the twisted $U_{\mathcal{B}\mathcal{J}\mathcal{E}\mathcal{J}}(\mathbf{L}^{(6)})$ (correspondingly $U_{\mathcal{B}\mathcal{J}\mathcal{E}\mathcal{J}}(so(5))$) with the costructure defined by the relations

$$\begin{aligned} \Delta_{\mathcal{B}\mathcal{J}\mathcal{E}\mathcal{J}}(H_{1+2}) &= H_{1+2} \otimes e^{-\sigma_{1+2}} + 1 \otimes H_{1+2} + E_2 \otimes E_1 e^{-\frac{3}{2}\sigma_{1+2} - \frac{1}{2}\sigma_{\mathcal{B}\mathcal{O}}} - \frac{1}{2}H_{1-2} \otimes E_1^2 e^{-2\sigma_{1+2} - \sigma_{\mathcal{B}\mathcal{O}}} \\ \Delta_{\mathcal{B}\mathcal{J}\mathcal{E}\mathcal{J}}(H_{1-2}) &= H_{1-2} \otimes e^{-\sigma_{\mathcal{B}\mathcal{O}}} + 1 \otimes H_{1-2} \\ \Delta_{\mathcal{B}\mathcal{J}\mathcal{E}\mathcal{J}}(E_{1+2}) &= E_{1+2} \otimes e^{\sigma_{1+2}} + 1 \otimes E_{1+2} \\ \Delta_{\mathcal{B}\mathcal{J}\mathcal{E}\mathcal{J}}(E_2) &= E_2 \otimes e^{-\frac{1}{2}\sigma_{1+2} - \frac{1}{2}\sigma_{\mathcal{B}\mathcal{O}}} + 1 \otimes E_2 - H_{1-2} \otimes E_1 e^{-\sigma_{1+2} - \sigma_{\mathcal{B}\mathcal{O}}} \\ \Delta_{\mathcal{B}\mathcal{J}\mathcal{E}\mathcal{J}}(E_1) &= E_1 \otimes e^{\frac{1}{2}\sigma_{1+2} + \frac{1}{2}\sigma_{\mathcal{B}\mathcal{O}}} + e^{\sigma_{1+2}} \otimes E_1 \\ \Delta_{\mathcal{B}\mathcal{J}\mathcal{E}\mathcal{J}}(E_{1-2}) &= E_{1-2} \otimes e^{\sigma_{\mathcal{B}\mathcal{O}}} + 1 \otimes E_{1-2} - E_1 \otimes E_1 e^{-\frac{1}{2}\sigma_{1+2} + \frac{1}{2}\sigma_{\mathcal{B}\mathcal{O}}} - \frac{1}{2}E_{1+2} \otimes E_1^2 e^{-\sigma_{1+2}} \\ \Delta_{\mathcal{B}\mathcal{J}\mathcal{E}\mathcal{J}}(E_{\mathcal{B}\mathcal{O}}) &= E_{\mathcal{B}\mathcal{O}} \otimes e^{\sigma_{\mathcal{B}\mathcal{O}}} + 1 \otimes E_{\mathcal{B}\mathcal{O}}. \end{aligned}$$

The twisted coproducts for the other four generators of $U_{\mathcal{B}\mathcal{J}\mathcal{E}\mathcal{J}}(so(5))$ can be expressed as finite sums of the $so(5)$ generators and the exponentials of σ_{1+2} and $\sigma_{\mathcal{B}\mathcal{O}}$. Using a particular set of generators nonlinearly related to the undeformed ones, it can be demonstrated that the bialgebra structure of $U_{\mathcal{B}\mathcal{J}\mathcal{E}\mathcal{J}}(so(5))$ coincides with the one presented in [9], where it was obtained as a direct solution of the conditions of coassociativity while the twisted character of the deformation was not studied.

The Jordanian twists on the deformed carrier spaces found above will give rise to different new constructions, enlarging the list of explicit solutions of the Yang–Baxter equation, deformed Yangians and integrable models. In particular, due to the embedding of the simple Lie algebras g into the corresponding Yangians (as Hopf subalgebras) $U(g) \subset \mathcal{Y}(g)$ [1] the Yangian R -matrix can be twisted by the same \mathcal{F} as defined for g . As a result for the case of orthogonal algebra $g = so(M)$ the R -matrix of $\mathcal{Y}(g)$ (in the defining representation $\rho \subset \text{Mat}(M, C) \otimes \text{Mat}(M, C)$) can be changed:

$$u\rho(1 \otimes 1) + \mathcal{P} - \frac{u}{u - 1 + M/2}\mathcal{K} \longrightarrow u\rho(\mathcal{F}_{21}\mathcal{F}^{-1}) + \mathcal{P} - \frac{u}{u - 1 + M/2}\rho(\mathcal{F}_{21})\mathcal{K}\rho(\mathcal{F}^{-1}).$$

(Here u is a spectral parameter and the operator \mathcal{K} that is obtained from \mathcal{P} by transposing its first tensor factor.) Henceforth the density of the integrable spin chain Hamiltonian is changing as well: $\mathcal{P}\rho(\mathcal{F}_{21}\mathcal{F}^{-1}) + \frac{1}{1-M/2}\rho(\mathcal{F})\mathcal{K}\rho(\mathcal{F}^{-1})$ (cf the $sl(2)$ -case [10]).

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